

Error estimation in homogenisation

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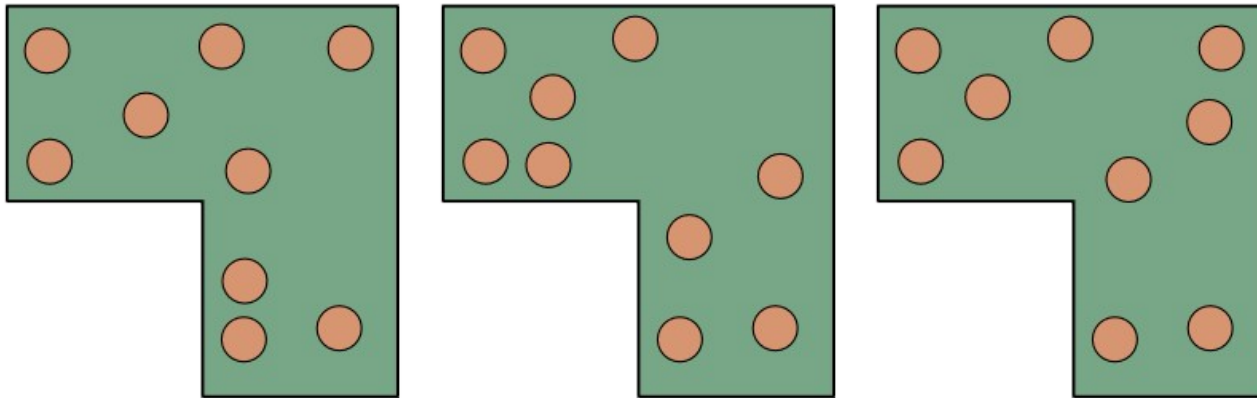
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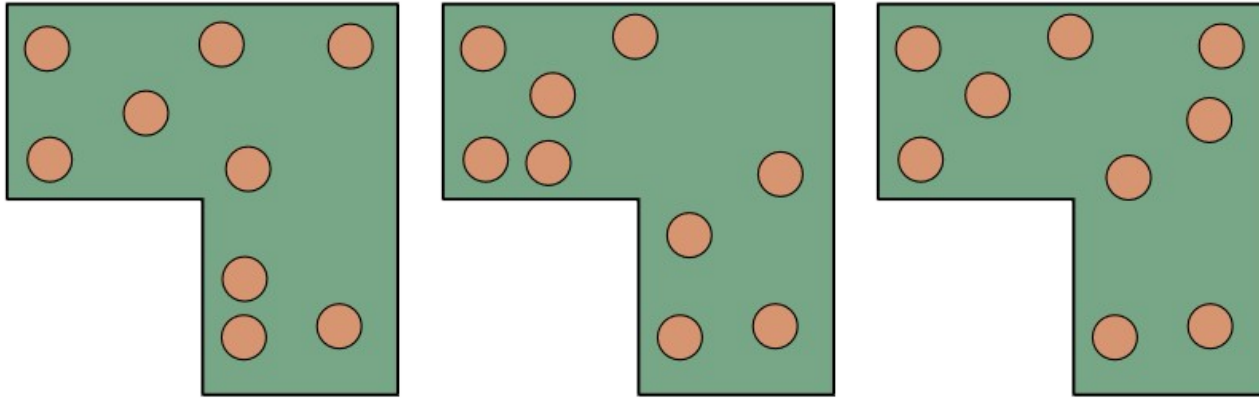
Motivation

Problem: Analysis of an heterogeneous materials. Vague information available. The position of the particles is not available.

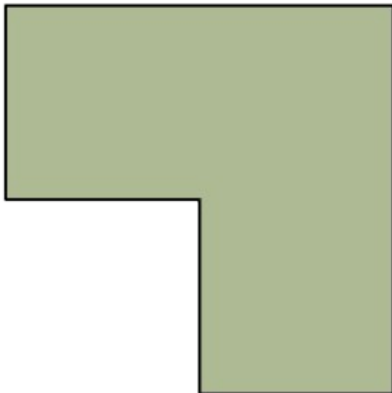


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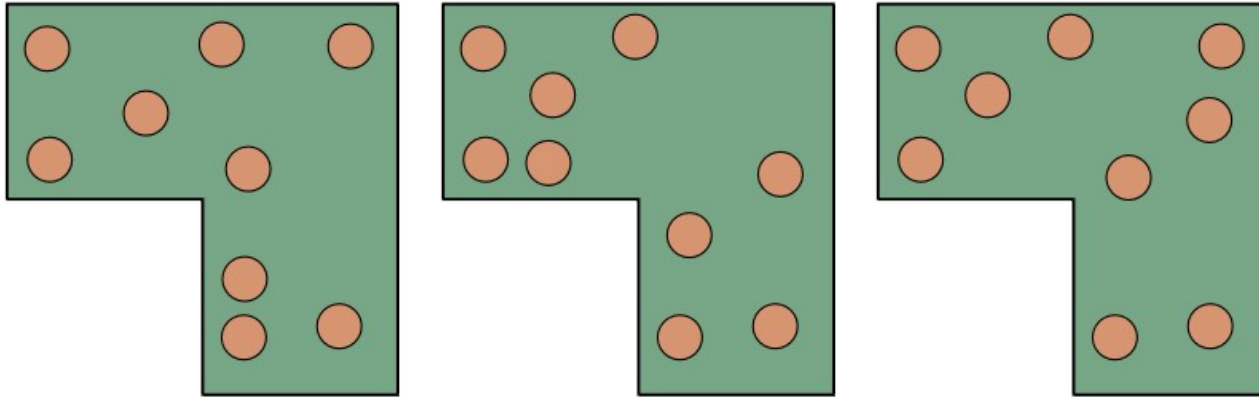


Solution: Homogenisation.

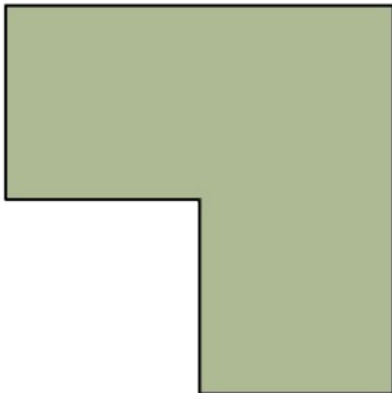


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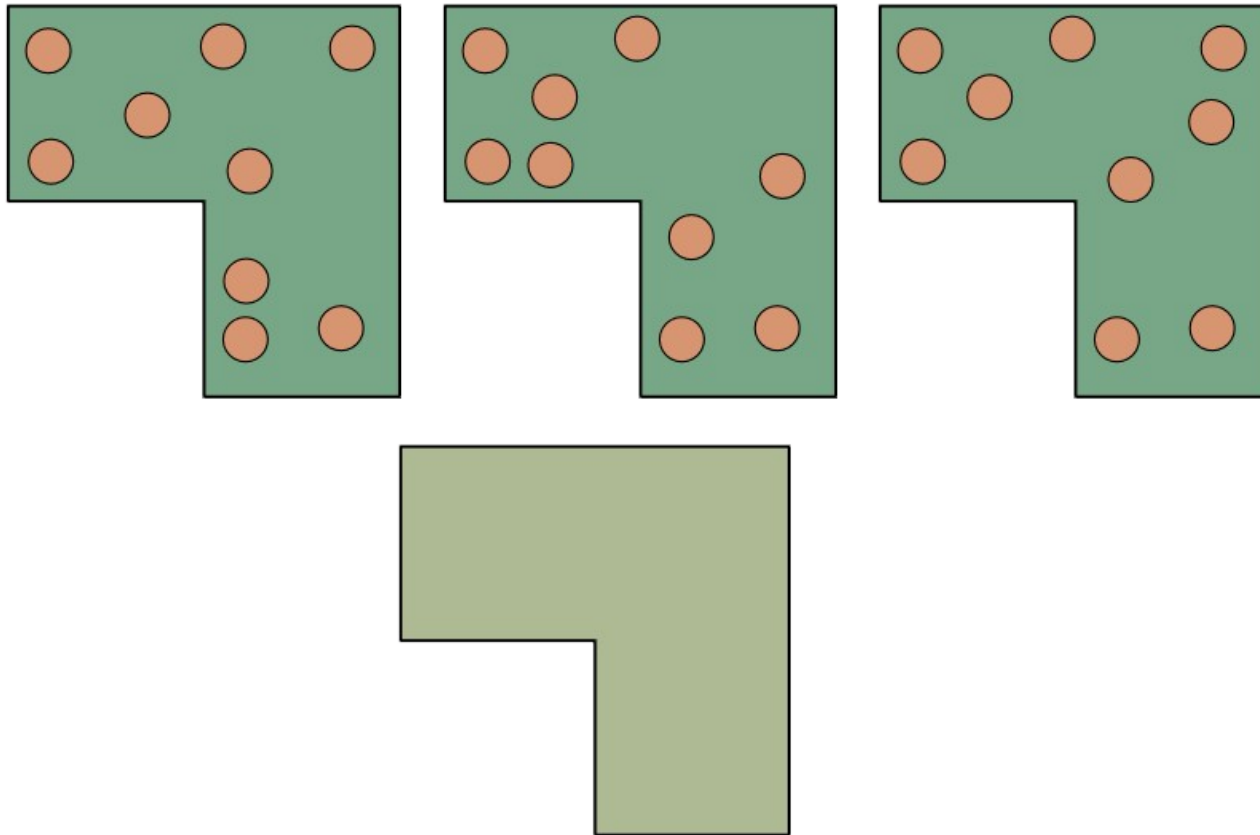
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New problem:
Assess the validity of the
homogenisation.

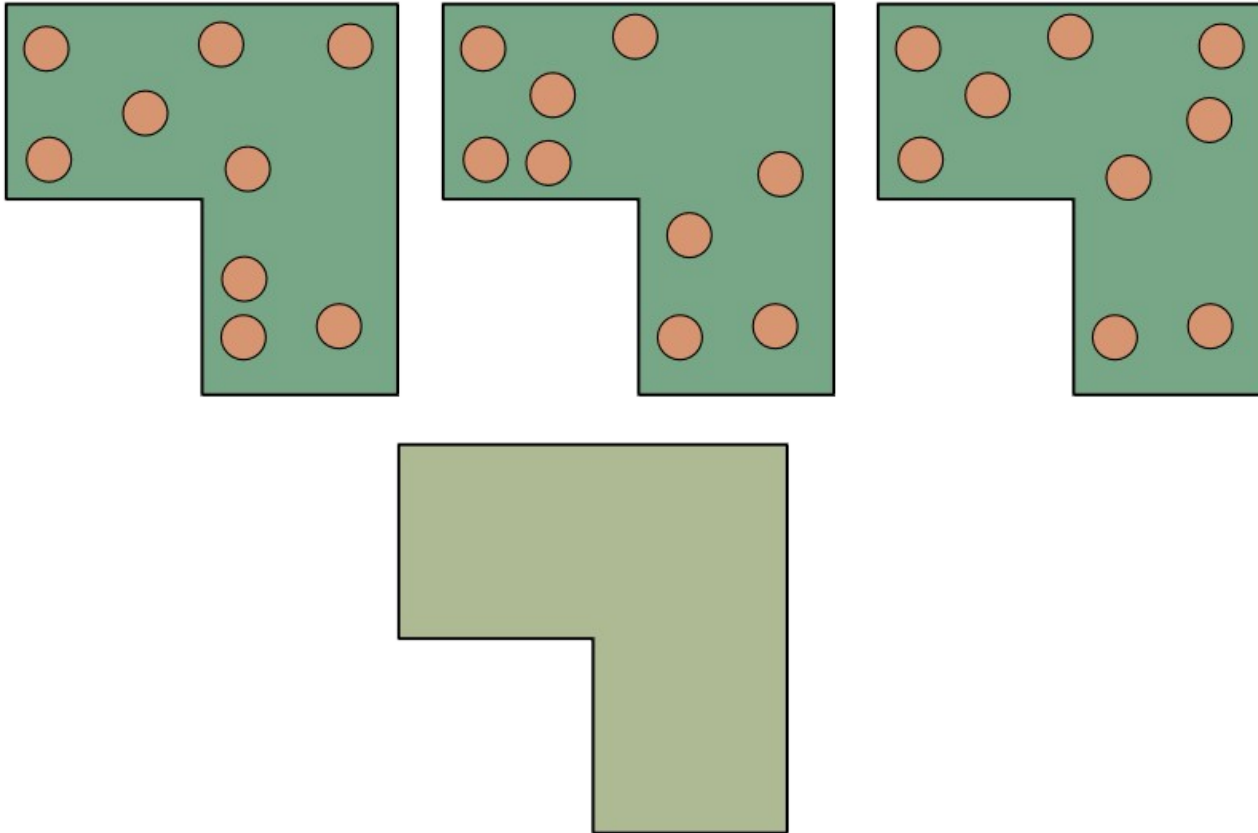
Proposed solution

Idea: Understand the original problem as an SPDE (the center of particles is a random variable) and bound the distance between both models



Proposed solution

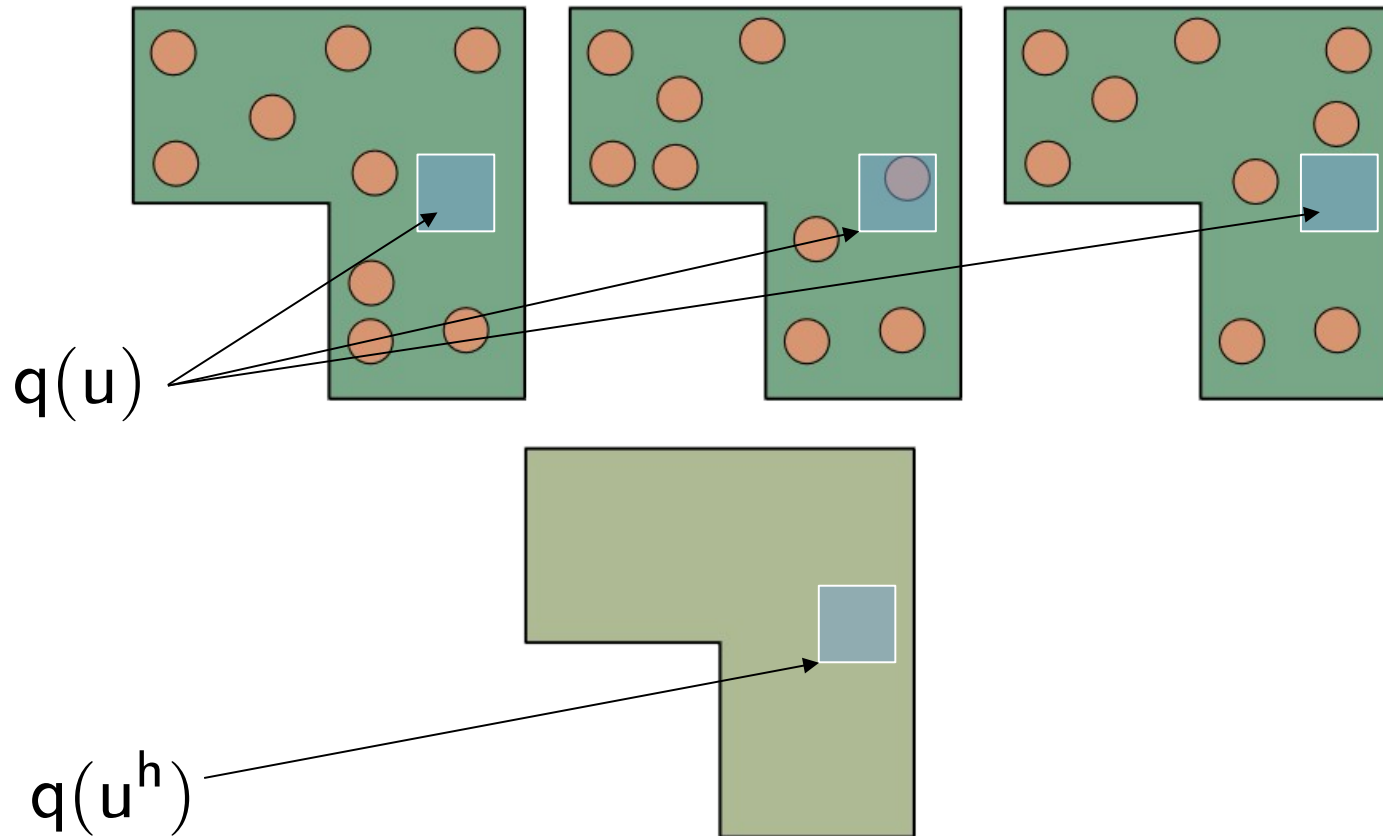
SPDE: Stochastic partial differential equation.
Collection of parametric problems + probability density function



Proposed solution

QoI: Quantity of interest. The output. Scalar that depends of the solution.

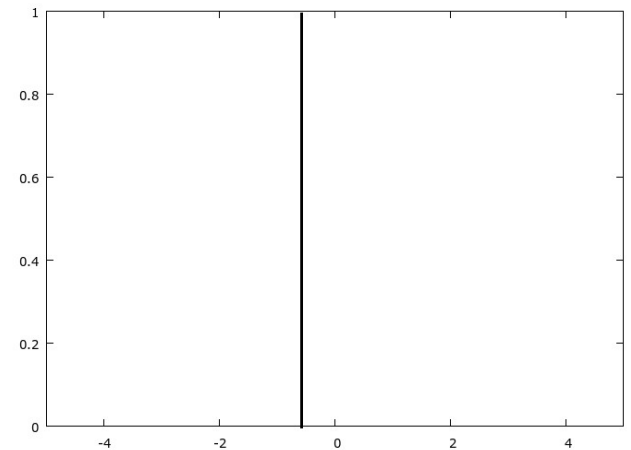
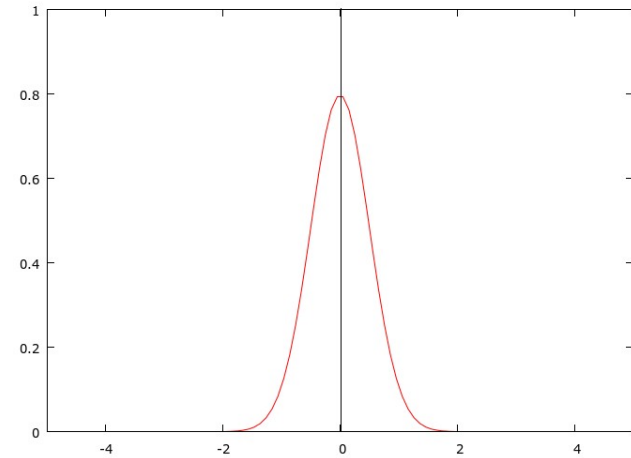
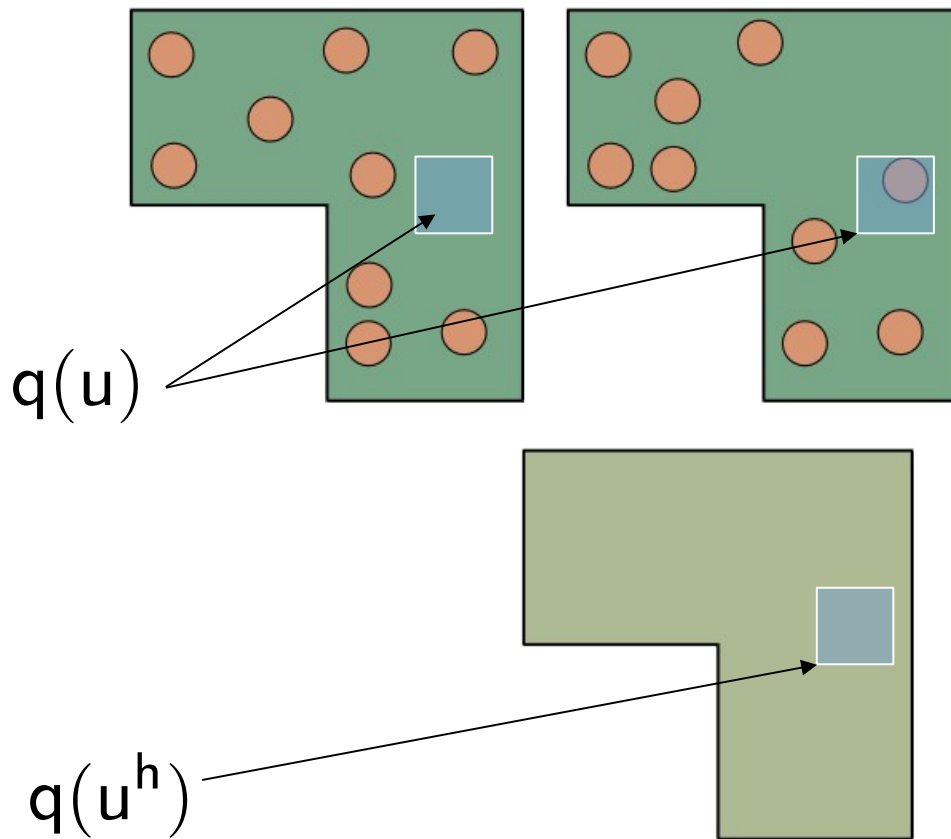
$$q(u) = \int_{\Omega} \int_{\Theta} \gamma(x) \cdot u(x, \theta) \quad (\text{linear})$$



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Heat equation

Heterogeneous problem

$$a(u, v) = \int_{\Omega} \int_{\Theta} k \nabla u \cdot \nabla v$$

$$l(v) = \int_{\Omega} \int_{\Theta} f v - \int_{\partial\Omega} g v$$

$$a(u, v) = l(v) \quad \forall v \in V$$

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Homogeneous problem

$$a_0(u_0, v) = \int_{\Omega} k_0 \nabla u \cdot \nabla v$$

$$a_0(u_0, v) = l(v) \quad \forall v \in V_0$$

$$a_0(u^h, v) = l(v) \quad \forall v \in V_0^h \subseteq V_0$$

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Aim: Bound

$$q(u) - q(u^h)$$

The computation of the bound must be deterministic.

Hypothesis

Deterministic boundary conditions

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Constant volume fraction

$$\int_{\Omega} k(x, \theta) = v_f k_I + (1 - v_f) k_M \quad \forall \theta \in \Theta$$

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Constant PDF over the domain

$$\underbrace{\int_{\Theta} k(x, \theta)}_{E[k]} = v_f k_I + (1 - v_f) k_M \quad \forall x \in \Omega$$

The unknown is the flux field and

$$\nabla \cdot \hat{\mathbf{Q}} = f \quad x \in \Omega$$

$$\hat{\mathbf{Q}} \cdot \mathbf{n} = g \quad x \in \partial\Omega_N$$

are fulfilled strongly.

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In order to derive bounds, we will also need to use an homogenised “flux” FE solution \hat{Q}

Error in the energy norm

Rewriting the problem in terms of the flux and the temperature

$$\nabla \cdot \mathbf{Q} = f \quad \forall x \in \Omega \times \Theta$$

$$\mathbf{Q} \cdot \mathbf{n} = g \quad \forall x \in \partial\Omega_N \times \Theta$$

$$u = h \quad \forall x \in \partial\Omega_D \times \Theta$$

$$\mathbf{Q} + k\nabla u = 0 \quad \forall x \in \Omega \times \Theta$$

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$\hat{\mathbf{Q}}$ will fulfill exactly the first 2 equations.

u^h will fulfill exactly the 3rd equation.

In general, $\hat{\mathbf{Q}} + k\nabla u^h \neq 0$ Discrepancy = measure of the error

Error in the energy norm

Formalizing this idea, it can be shown that

$$\|u - u^h\| = \|e\| \leq \|\hat{Q} + k\nabla u^h\|_{k-1} =: \eta$$

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Expanding η^2

$$\begin{aligned} \eta^2 &= \int_{\Omega} \int_{\Theta} k^{-1} \hat{Q} \cdot \hat{Q} + \int_{\Omega} \int_{\Theta} k \nabla u^h \cdot \nabla u^h + 2 \int_{\Omega} \int_{\Theta} \hat{Q} \cdot \nabla u^h \\ &= E[k^{-1}] \int_{\Omega} \hat{Q} \cdot \hat{Q} + E[k] \int_{\Omega} \nabla u^h \cdot \nabla u^h + 2 \int_{\Omega} \hat{Q} \cdot \nabla u^h \end{aligned}$$

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Deterministic quantity

Error in the quantity of interest

The error in energy norm is not always relevant.

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Cauchy-Schwarz inequality

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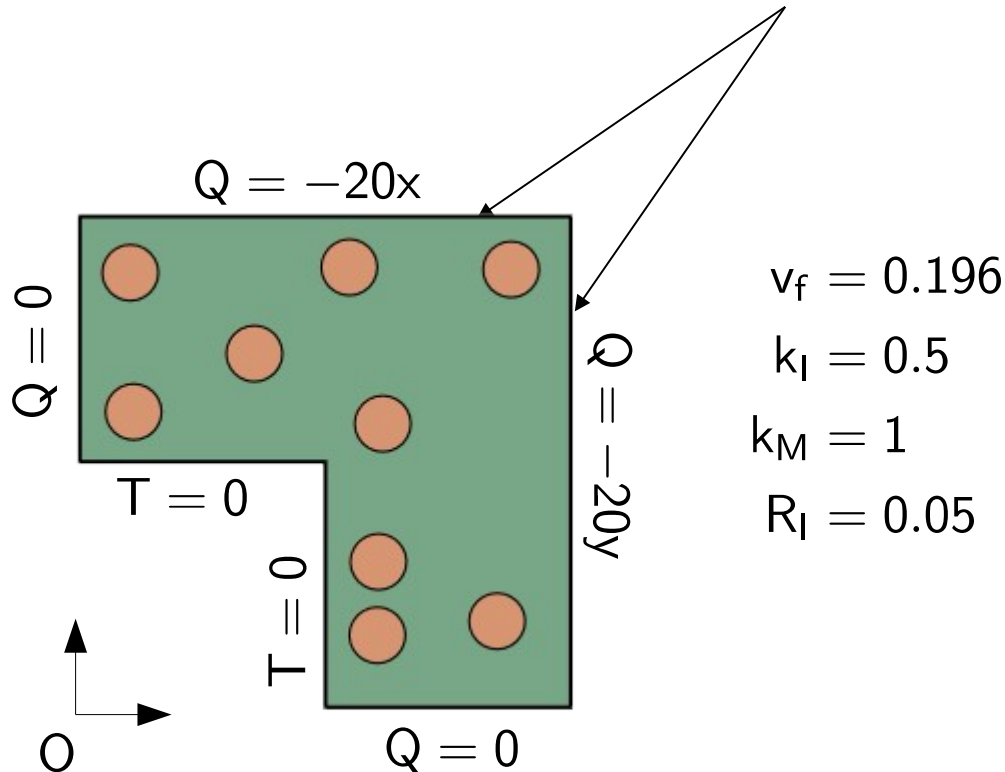
$$|a(e_\phi, e)| \leq \|e_\phi\| \|e\|$$

Use the bound in the energy norm,

$$R(\phi^h) - \eta\eta_\phi \leq q(u) - q(u^h) \leq R(\phi^h) + \eta\eta_\phi$$

Validation

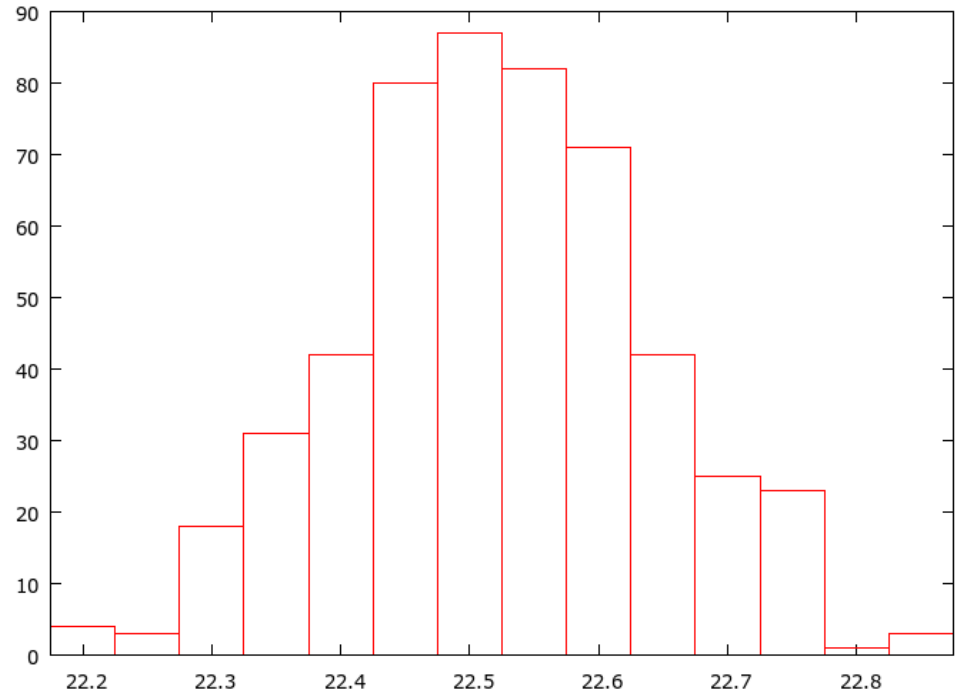
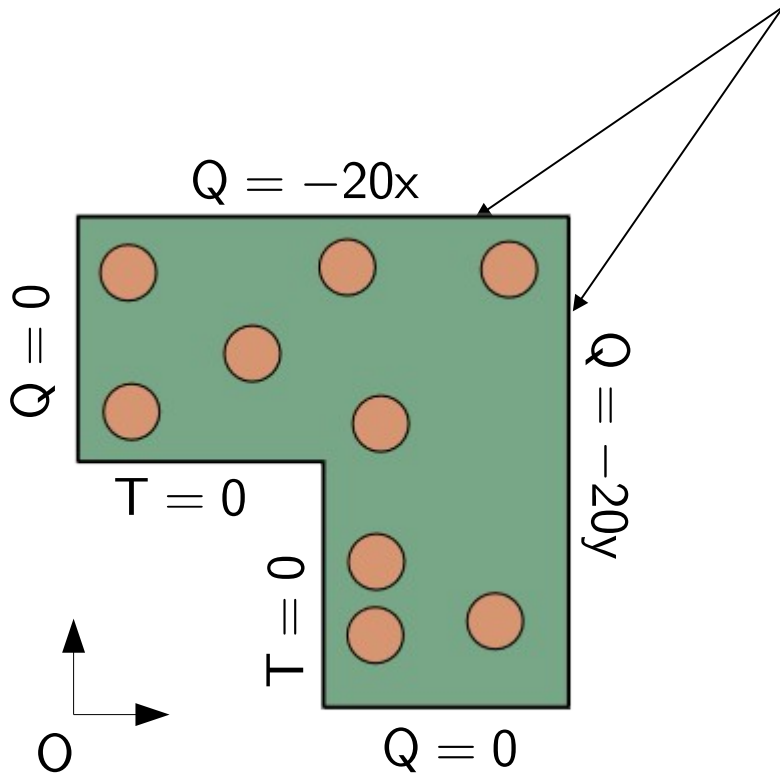
The quantity of the interest is the average temperature in the exterior faces.



The “exact” quantity of interest is computed with 512 MC realisations.

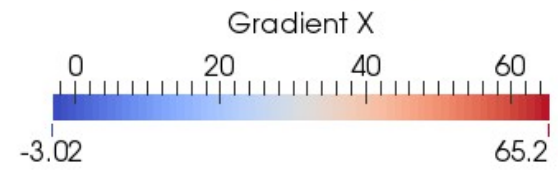
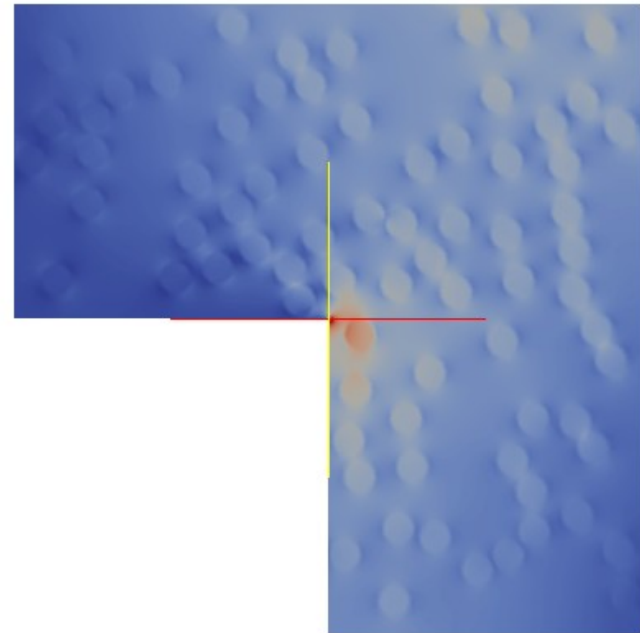
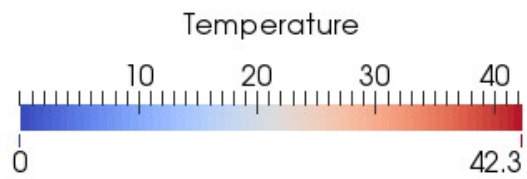
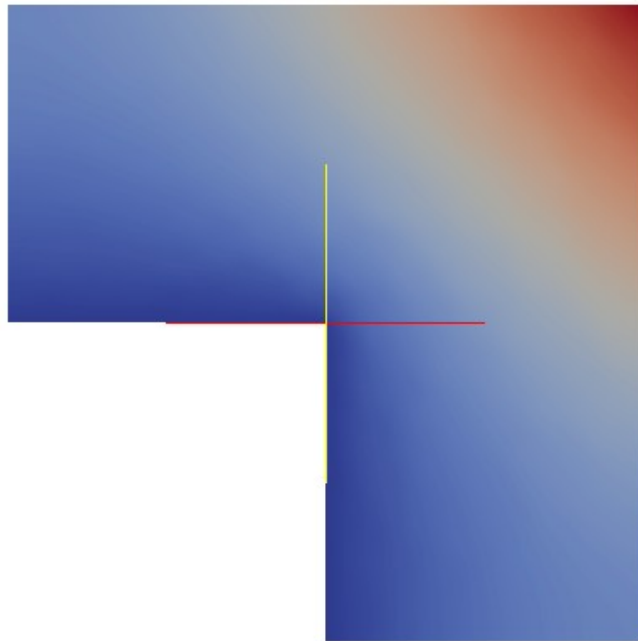
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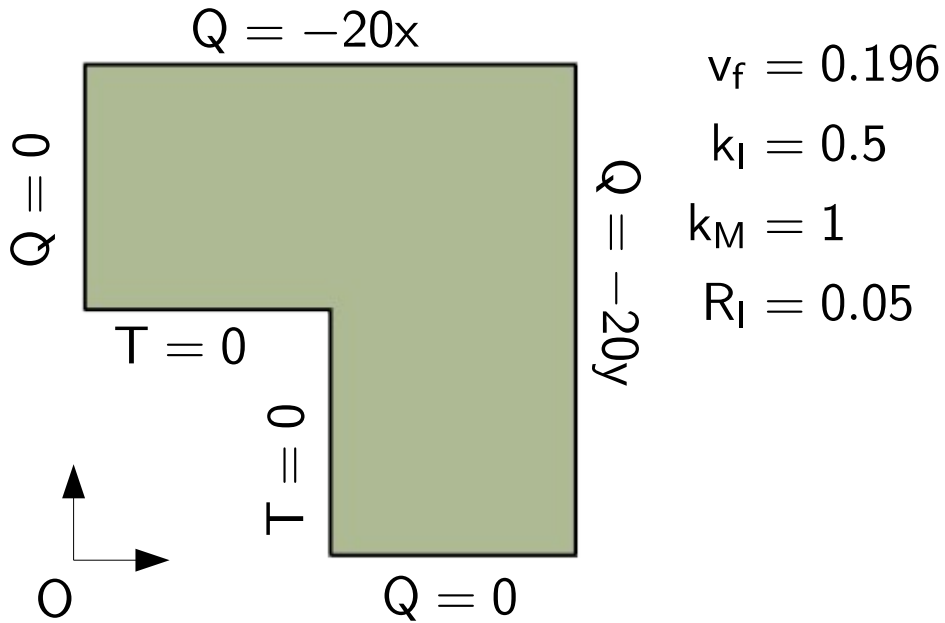
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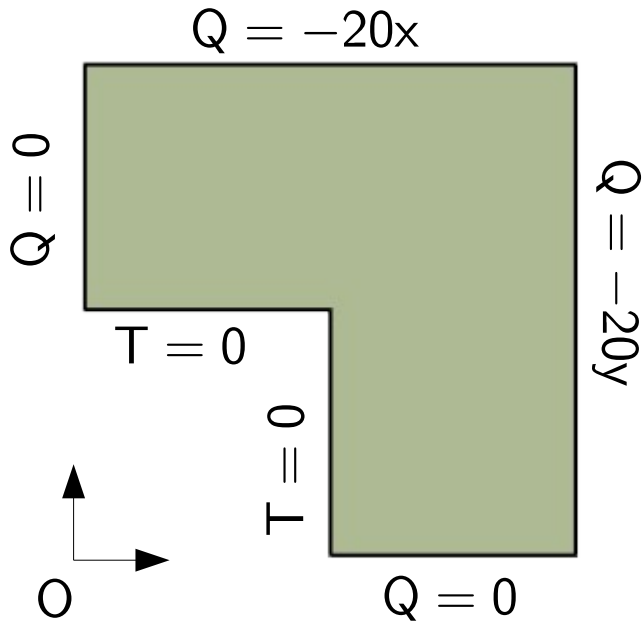
Validation

Studied in a domain homogenised through rule of mixture.



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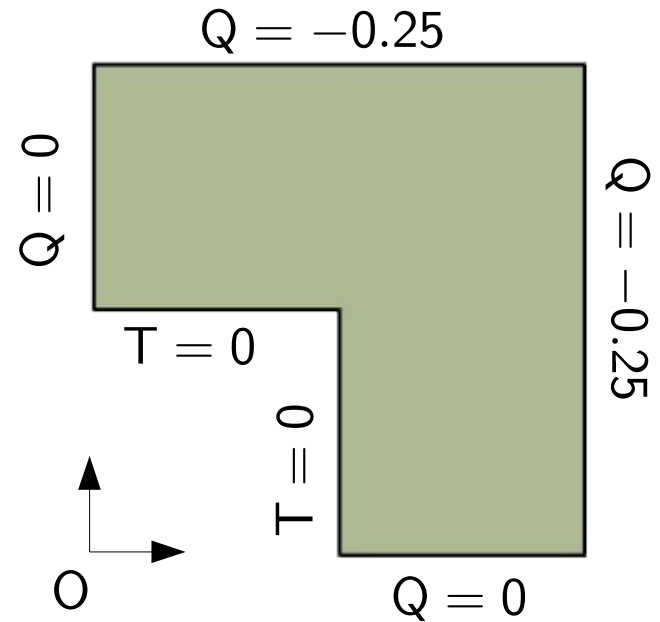
$$k_l = 0.5$$

$$k_M = 1$$

$$R_l = 0.05$$

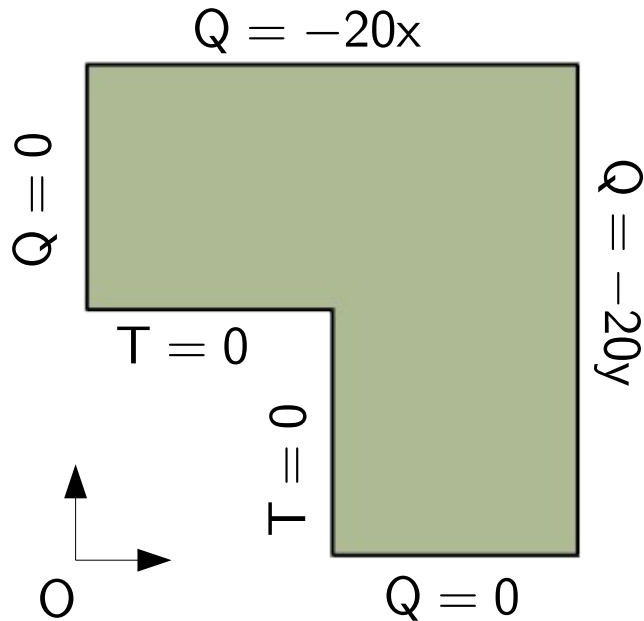
Dual problem

$$a_0(\phi^h, v) = q(v)$$



Validation

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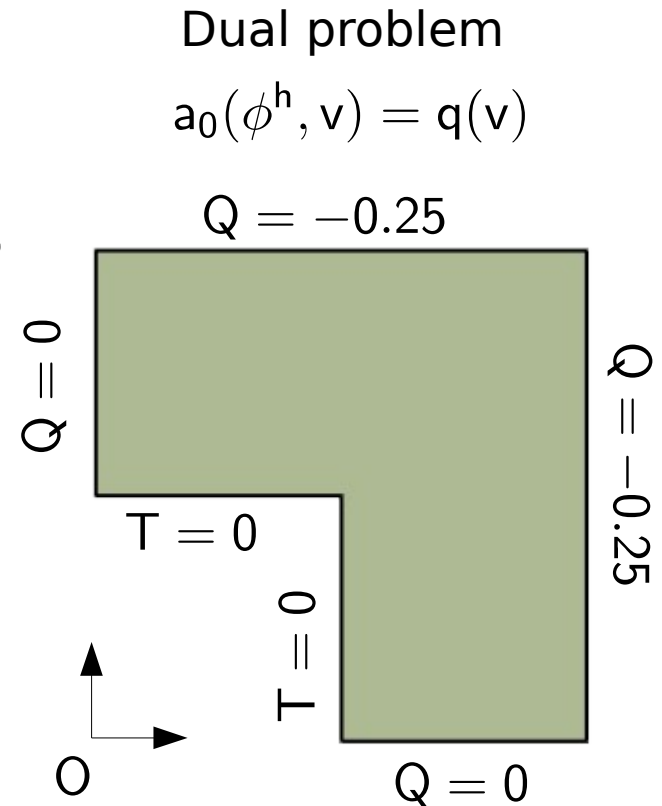


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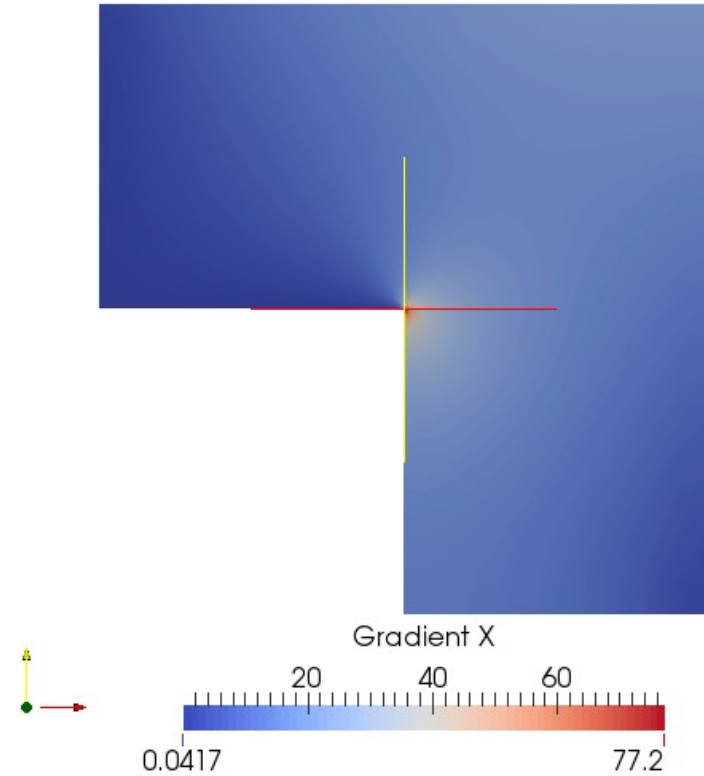
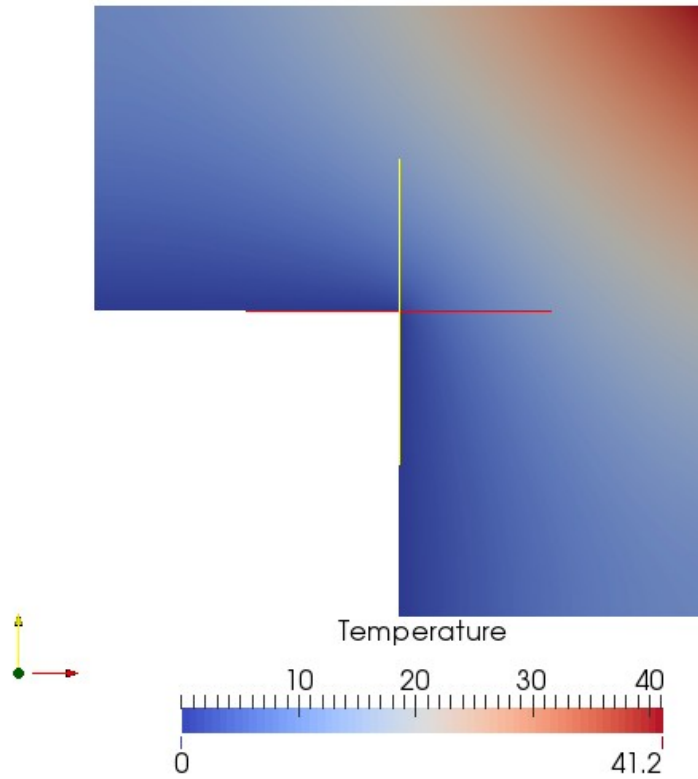
$$R_l = 0.05$$



Two problems solved twice:

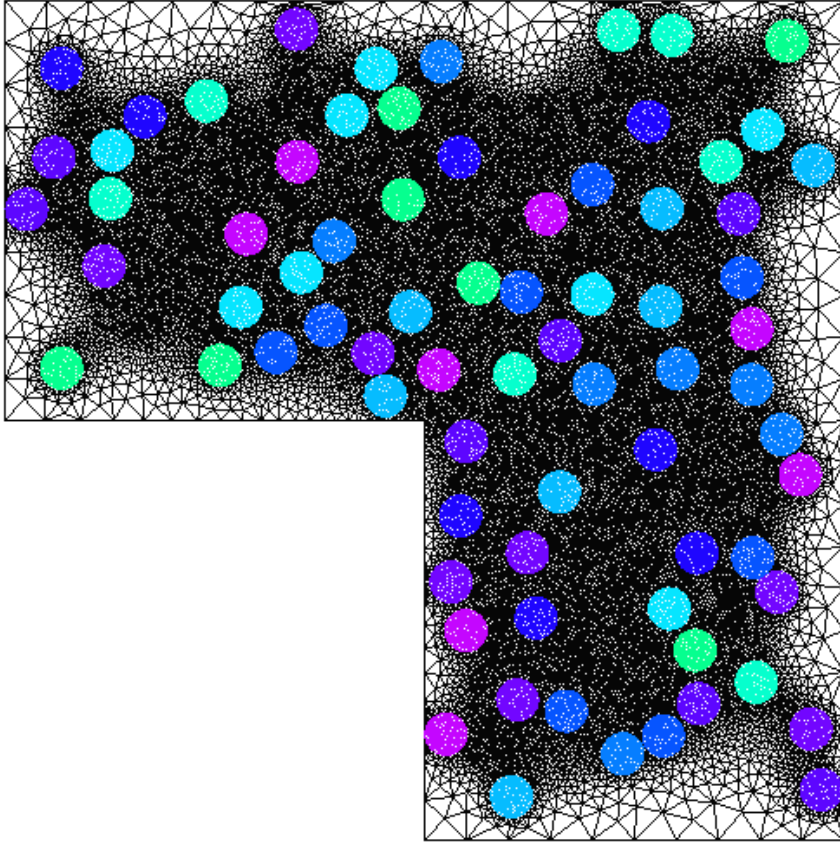
- Using "temperature" FE u^h, ϕ^h
- Using "flux" FE \hat{Q}, \hat{Q}_ϕ

Validation

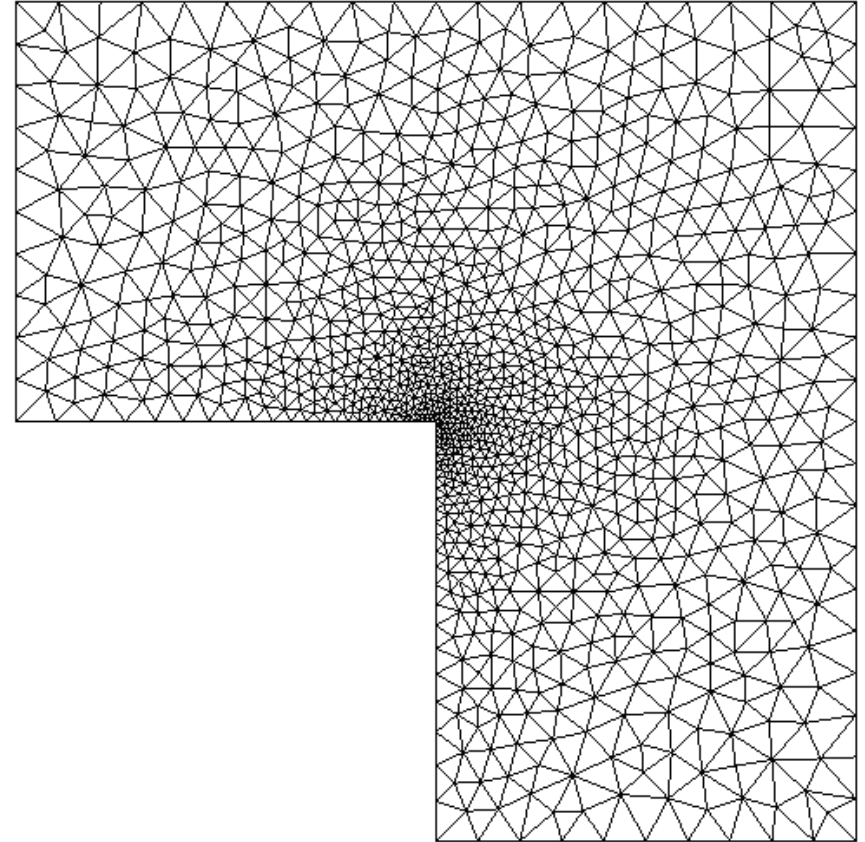


$q(u^h)$	$ q(u) - q(u^h) $	$\leq \zeta_u$	$\zeta_l + q(u^h) \leq$	$q(u)$	$\leq \zeta_u + q(u^h)$
21.92	0.63	1.84	20.08	22.55	23.76

Validation



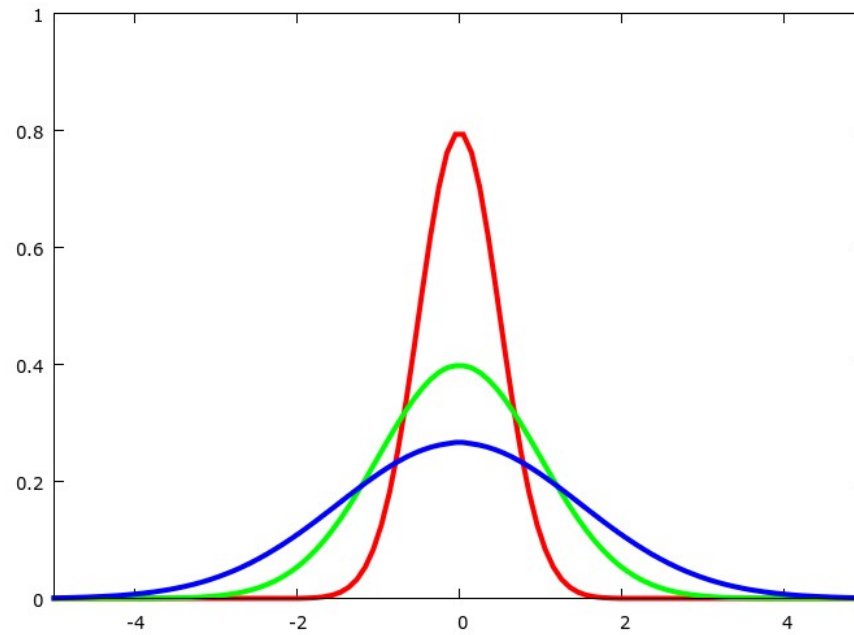
Around 60 000 elements
512 problems, 512 different meshes
Full PDF



Around 2000 elements
4 problems, 1 mesh
Bounds on the expectation

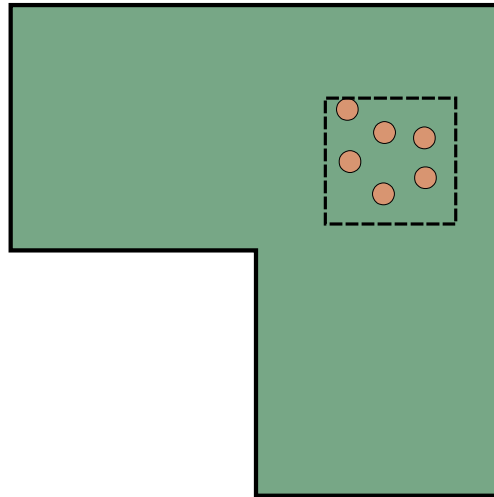
Work in progress

- A bound on the variance.



Work in progress

- Enhanced model. Insert patches with particles in parts of the domain.



Summary

- A bound for the homogenisation error was presented.
- The computation of the bound is deterministic.
- The error estimate, should be used with care when there is a high contrast between the material properties.

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Thank you for your attention.